MATH-F410 – REP. DES GROUPES & APP. A LA PHYS.

Solutions des exercices 2022-2023

Séance 4 : Application des groupes finis à l'algèbre de Dirac

1. There is a single irreducible representation of the Dirac algebra

In this exercise, it is important to differentiate when we are dealing with properties particular to a certain representation (like the one given in the question), and when we are being general and making statements that only depend on the structure provided by the Dirac algebra. In the following sections, and until we explicitly say otherwise, we are being **completely general**. This means that we will only use the structure provided by the Dirac algebra (anticommutators of the four objects γ^μ, $\mu = 0, 1, 2, 3$) to prove our results. This guarantees that the results do not depend on the particular representation of the γ^μ. Notice that the defining anticommutation relations guarantee that:

$$\left(\gamma^0\right)^2 = -I \ , \qquad \left(\gamma^i\right)^2 = I \ , \qquad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \text{if } \mu \neq \nu \ .$$

In the previous expressions and in what follows, $i, j, \ldots$ are spatial indices running from 1 to 3.

Section a)

This is just a case by case check. All the indices appearing in the following expressions are assumed to be different, and we use the known squares and anticommutation relations:

$$\begin{aligned}
(\pm I)^2 &= I \ , \\
(\pm\gamma^\mu)^2 &= \pm I \quad [+ \text{ if } \mu = i, - \text{ if } \mu = 0] \ , \\
(\pm\gamma^\mu\gamma^\nu)^2 &= -\left(\gamma^\mu\right)^2\left(\gamma^\nu\right)^2 = \pm I \ , \\
(\pm\gamma^\mu\gamma^\nu\gamma^\rho)^2 &= -\left(\gamma^\mu\right)^2\left(\gamma^\nu\right)^2\left(\gamma^\rho\right)^2 = \pm I \ , \\
\left(\pm\gamma^5\right)^2 &= \left(\gamma^0\right)^2\left(\gamma^1\right)^2\left(\gamma^2\right)^2\left(\gamma^3\right)^2 = -I \ .
\end{aligned}$$

In the end, we are just using that all different γ's anticommute and their squares are always $\pm I$. This shows that every element $a \in G$ has an inverse $a^{-1} = \pm a$.

Section b)

The first statement is just the formal version of saying that all elements either commute or anticommute. This is a consequence of different gamma matrices anticommuting, and equal ones commuting. We can therefore always move elements freely to left or right, at the cost of a $\pm$ sign. Thus, $ab = \pm ba$ for all elements $a, b \in G$. For the second part, for a given $a \in G$ different from (plus or minus) the identity), we must find an element $b \in G$ that anticommutes with it. This is done as follows:

- If $a = \pm\gamma^\mu$, pick a different gamma matrix $b = \gamma^\nu$ with $\nu \neq \mu$. This anticommutes with a.

- If $a = \pm\gamma^\mu\gamma^\nu$, pick $b = \gamma^\mu$. This anticommutes with γ^ν but commutes with γ^μ, so it anticommutes with a.

- If $a = \pm\gamma^\mu\gamma^\nu\gamma^\rho$, pick $b = \gamma^\sigma$ with $\sigma \neq \mu, \nu, \rho$ (this is possible because there are 4 different gammas!). This anticommutes with all the three gammas in the definition of a, so it anticommutes with a.

- Finally, if $a = \pm\gamma^5$, just pick any γ^μ, say $b = \gamma^0$. This anticommutes with $\gamma^1, \gamma^2, \gamma^3$ but commutes with itself, so it anticommutes with $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$.

We see that for any element in G different from $\pm I$, we can always find a gamma matrix that anticommutes with it. We will put this to good use later on.

Section c)

First of all, the identity is a member of G by construction. The inverse of any element $a \in G$ is another element of G, since we showed that $a^{-1} = \pm a$ and all the elements of this form are in G. We only have to check that G is closed under multiplication. This is again immediate from the anticommutators:

- If we multiply $\pm I$ times $a \in G$, we get $\pm a \in G$ by definition of G.

- If we multiply two elements different from (plus or minus) the identity, then we are multiplying two chains of gamma matrices. Anticommuting and writing squares as (plus or minus) the identity we can reduce this to a single, ordered chain of gamma matrices, which belongs to G by definition. Some examples:

$$\left(\gamma^0\gamma^1\gamma^2\right)\left(\gamma^1\gamma^3\right) = -\gamma^0\gamma^2\gamma^3 \in G\ , \qquad \left(\gamma^0\gamma^1\gamma^2\gamma^3\right)\left(\gamma^1\gamma^3\right) = -\gamma^0\gamma^1 \in G\ .$$

Section d)

First, notice that in this section **we work in the representation presented in the question**. The key point is what's known in the QFT literature as *trace technology*, essential to do computations involving Dirac fermions. Two immediate results are:

$$\chi\left(\pm I_4\right) = \pm 4\ , \qquad\qquad \chi\left(\pm\gamma^\mu\right) = 0\ .$$

Then (in what follows, all indices are different so we freely anticommute):

$$\chi\left(\gamma^\mu\gamma^\nu\right) = \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = -\mathrm{Tr}\left(\gamma^\nu\gamma^\mu\right) = -\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) \Longrightarrow \chi\left(\gamma^\mu\gamma^\nu\right) = 0\ ,$$
$$\chi\left(\gamma^5\right) = \mathrm{Tr}\left(\gamma^0\gamma^1\gamma^2\gamma^3\right) = -\left(\gamma^1\gamma^2\gamma^3\gamma^0\right) = -\mathrm{Tr}\left(\gamma^0\gamma^1\gamma^2\gamma^3\right) = -\chi\left(\gamma^5\right) \Longrightarrow \chi\left(\gamma^5\right) = 0\ .$$

This just uses anticommutation and ciclicity of the trace. When we have three gamma matrices we have to introduce an extra, different one, using that it squares to $\pm I$:

$$\chi\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) = \pm\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\left(\gamma^\sigma\right)^2\right) = \mp\mathrm{Tr}\left(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = \mp\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\left(\gamma^\sigma\right)^2\right) = -\chi\left(\gamma^\mu\gamma^\nu\gamma^\rho\right)\ .$$

So, once again, $\chi\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) = 0$. The only non-vanishing characters are those of $\pm I$, from which we conclude, noting that the group has 32 elements:

$$(\chi|\chi) = \frac{1}{32}\left(4^2 + (-4)^2\right) = 1 \ ,$$

so the representation is indeed irreducible.

Section e)

We can now go back to a general reasoning, independent of the representation. Clearly, $\pm I$ commutes with everything, so they form single-element conjugacy classes: $[I] = \{I\}$ and $[-I] = \{-I\}$. Furthermore, from b) we know that no other element can form a single-element conjugacy class, because for any $a \in G$ different from $\pm I$ we have some $b \in G$ such that $ba = -ab \Leftrightarrow bab^{-1} = -a$. Thus, $-a \in [a]$. This is the only other element (also due to b)), since for any $b \in G$ we know that $ba = \pm ab$, so $bab^{-1} = \pm a$. To sum up:

- We have 2 single element conjugacy classes, $[I]$ and $[-I]$.
- The remaining elements arrange themselves into two-element conjugacy classes, $[a] = \{a, -a\}$. Since there are 32 elements in total, we have 15 of these conjugacy classes.

All in all, there are 17 conjugacy classes.

Section f)

The number of irreps coincides with the number of conjugacy classes, so we have 17. From Burnside's theorem we can obtain the dimensions. We use the already known results that we have a 1-dimensional irrep (the trivial one) and a 4-dimensional one (the one provided in the question):

$$32 = 1 + 4^2 + \sum_{i=1}^{15} n_i^2 \ .$$

This equation can only be solved if all remaining irreps are 1-dimensional. So, to say it again, there are 16 1-dimensional irreps of the group G and a 4-dimensional one.

Section g)

We proceed in full generality, ignoring the already known dimensions of the irreps (we provide an argument for a general m-dimensional irrep). Let T be an m-dimensional irrep then. Notice that, for any $g \in G$:

$$T(g)T(-I) = T(-g) = T(-Ig) = T(-I)T(g) \ .$$

So $T(-I)$ commutes with every representative $T(g)$ of the irrep. This shows that it can be thought of as an entanglement operator, and by Schur's lemma it must be either 0 (which is not possible because by definition of a representation $T(-I)$ is an invertible operator) or $T(-I) = \lambda I_m$ (subindex m here emphasizes the dimension of the irrep). This is a very important lesson to keep, so let us repeat it: any non-zero operator that commutes with every representative of an irrep must act as a multiple of the identity in the space on which the irrep acts. In this case, we can deduce something more about λ as follows:

$$T(-I)T(-I) = T(I) = I_m \Rightarrow \lambda^2 I_m = I_m \Rightarrow \lambda = \pm 1 \ .$$

Section h)

We can now use the result in Section b) to show that in any representation with $T(-I) = I_m$ all the representatives commute:

$$T(a)T(b) = T(ab) = T(\pm ba) = T(\pm I)T(b)T(a) = T(b)T(a) \ .$$

From Schur's lemma, $T(a) = \lambda_a I_m$ for any $a \in G$. But then, since all representatives are multiples of the identity, any vector defines an invariant subspace: $T(a)v = \lambda_a v$. If T is irreducible the only possibility is that it is one-dimensional (otherwise there would be non-trivial invariant subspaces). Finally, since $a^2 = \pm I$, in this representation it is $1 = T(a^2) = T(a)^2 = \lambda_a^2$, so $\lambda_a = \pm 1$ and therefore $T(a) = \pm 1$.

Section i)

We prove the converse now: let T be a 1-dimensional representation, so that all representatives commute. Take $a, b \in G$ such that $ab = -ba$ (always possible due to Section b). Then:

$$T(b)T(a) = T(a)T(b) = T(ab) = T(-Iba) = T(-I)T(b)T(a) \ ,$$

from which it follows that $T(-I) = 1$.

Section j)

Let ρ_i be a 1-dimensional representation of the group G. We know that $\rho_i(I) = \rho_i(-I) = 1$. Any representative of a string of gamma matrices is determined by the representatives of the single gamma matrices, *e.g.*, $\rho_i(\gamma^\mu\gamma^\nu) = \rho_i(\gamma^\mu)\rho_i(\gamma^\nu)$. Thus, the representation is fully determined once we fix $\rho_i(\gamma^\mu)$ for the 4 different values of μ (notice that $\rho_i(-a) = \rho_i(a)$). Section h) shows that $\rho_i(\gamma^\mu) = \pm 1$, so we have $2^4 = 16$ different ways to assign values to $\rho_i(\gamma^\mu)$. They give different irreps since the characters are different (the γ^μ are in different conjugacy classes for different values of μ). These choices then determine the 16 1-dimensional irreps of G: $\rho_i(\gamma^\mu) = 1$ for all μ gives the trivial representation, and then the remaining 15 possibilities give the non-trivial ones.

Section k)

An m-dimensional representation of Dirac's algebra assigns a representative to each γ^μ, call it $t(\gamma^\mu)$, in a way that the anticommutation relations are satisfied. To extend it to a representation of the group G, which we call T, we must guarantee that products are respected. This leaves no choice on how to do it:

- Take $T(\gamma^\mu) = t(\gamma^\mu)$.

- Take $T(I) = I_m$ (this is forced for group representations, and also $t(\gamma^i)t(\gamma^i) = I_m = T(\gamma^i\gamma^i) = T(I)$ using algebra properties and the previous definition).
- Take $T(-I) = t(\gamma^0)t(\gamma^0) = -I_m$ (fixed by algebra properties).
- For any string of gamma matrices, take $T(\gamma^\mu \dots \gamma^\nu) = T(\gamma^\mu)\dots T(\gamma^\nu) = t(\gamma^\mu)\dots t(\gamma^\nu)$.
- Negatives are defined as $T(-a) = T(-I)T(a) = -T(a)$.

This procedure assigns an invertible operator to each element in G (all $t(\gamma^\mu)$ are invertible from the algebra properties, and therefore products of them are as well). It does it in a way that respects the G multiplication law by construction (all elements of G are strings of gamma matrices, and the previous construction guarantees that the homomorphism property is respected). We have then a valid representation of the group, T.

Section l)

Already the previous construction shows what's going to go wrong when going from group representations to algebra representations. The algebra representation demands $t(\gamma^0)t(\gamma^0) = -I_m$, and since $-I = (\gamma^0)^2$, the associated group representation must have $T(-I) = -I_m$. We saw that this isn't true in 1-dimensional representations. To make it more precise, in any 1-dimensional representation everything commutes. But Dirac's algebra demands $T(\gamma^\mu)T(\gamma^\nu) = -T(\gamma^\nu)T(\gamma^\mu)$ if $\mu \neq \nu$. So at least the 1-dimensional representations of G do not induce a corresponding representation of the algebra.

Section m)

This is just the final step of all the previous argument. Irreducibility is a criterion that is the same for group an algebra representations. If an algebra representation is irreducible (it leaves no subspace invariant), then the associated group representation is as well, because the algebra representatives are part of it. Conversely, if a group representation is irreducible, since all representatives are products of the basic ones for gamma matrices, it can't happen that all representatives $T(\gamma^\mu)$ leave a certain subspace invariant (otherwise all the representatives $T(a)$ would as well). Then, the associated algebra representation, if it exists, it is irreducible.

Let us put together this with the results from previous sections. Let t be any irreducible representation of Dirac's algebra. Then, it necessarily induces a representation T of the group G, and this is also irreducible. It cannot be one-dimensional due to Section l) (everything would commute otherwise), so from our knowledge of the irreps of G we conclude that t can only be the 4-dimensional irreducible representation given in the question. This is the only irrep of Dirac's algebra.